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COMMON FIXED POINT THEOREMS OF GREGUS TYPE IN CONVEX METRIC SPACES

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Abstract. In this paper, we prove common fixed point theorems of Gregus type for three mappings in convex metric spaces. We extend and generalize some well known results by many authors.

1. Introduction

The notion of convex metric spaces was initially introduced by Takahashi [17]. He and others gave some fixed point theorems for nonexpansive mappings in convex metric spaces ([2], [5], [7], [8], [9], [13], [15], [16]).

On the other hand Gregus [6] proved a fixed point theorem in Banach spaces, which is called Gregus fixed point theorem and then many authors have obtained some fixed point theorems of Gregus type.

The aim of this paper is to prove some common fixed point theorems of Gregus type for compatibile mappings in convex metric spaces.

Recently, Huang and Cho [8], proved common fixed point theorems of Gregus type in convex metric spaces. They proved results for two mappings. In this paper, we prove results for three mappings. We also extend and generalize some known results of Gregus type.

2. Preliminaries

In this section, we give some definitions and lemmas for our main results.

Definition 1. *Let (X,d) be a metric space and $J=[0,1]$. A mapping $W: X\times X\times J \to X$ is called a convex structure on X if for each $(x,y,\lambda)\in X\times X\times J$ and $u\in X$,*

$$d\big(u, W(x,y,\lambda)\big) \le \lambda d(u,x) + (1-\lambda) d(u,y).$$

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a metric space X together with a convex structure W is called a convex metric space.

Definition 2. *A nonempty subset K of X is said to be convex if,*

$$W(x, y, \lambda) \in K$$

For all $(x, y, \lambda) \in K \times K \times J$.

Obviously, a Banach space or any convex subset of a Banach space is a convex metric space. But there are many examples of convex metric spaces which are not embedded in any Banach spaces. For further information on convex metric spaces, we refer to [17].

Definition 3. *Let (X, d) be a convex metric space and let K be a convex subset of X. A mapping $S : K \to K$ is said to be W-affine if*

$$SW(x, y, \lambda) = W(Sx, Sy, \lambda) \quad \textit{for all} \quad (x, y, \lambda) \in K \times K \times J.$$

Definition 4. [10]. *Let (X, d) be a metric space and let S, $T : X \to X$ be two mappings. S and T are said to be compatible if, whenever $\{x_n\}$ is a sequence in X such that Sx_n, $Rx_n \to t \in X$, then*

$$d(STx_n, TSx_n) \to 0.$$

Lemma 1. [10]. *Let S and T be compatible mappings of a metric space (X, d) into itself. If $Sz = Tz$ for some $z \in X$, then $STz = TSz = S^2z = A^2z$.*

Lemma 2. [10]. *If S and T are compatible self maps of a metric space (X, d) and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X, then $\lim_{n\to\infty} TSx_n = Sz$ if S is continuous.*

Lemma 3. *Let $a > 0$, $c > 0$ and $p \geq 1$. If $a + c < (3 - 3^{1-p})(3^p - 1)^{-1}$, then $a^{1/p} + c^{1/p} < 1$.*

Proof. Let $f(x) = x^p$ for all x in $(0, \infty)$ and $p \geq 1$. It follows from $p \geq 1$ that

$$f\big((1/3)(x + y)\big) \leq (1/3)f(x) + (1/3)f(y)$$

for all $x, y > 0$. Thus we have

$$\big((1/3)(a^{1/p} + c^{1/p})\big)^p \leq (1/3)(a + c) < (1 - 3^{-p})(3^p - 1)^{-1},$$

which implies that

$$a^{1/p} + c^{1/p} < \big(3^p(1 - 3^{-p})(3^p - 1)\big)^{1/p} = \left(3^p\left(\frac{3^p - 1}{3^p} \cdot \frac{1}{3^p - 1}\right)\right)^{1/p}.$$

This completes the proof.

3. Main results

Throughout this section, we assume that X is a complete convex metric space with a convex structure W and K is a nonempty closed convex subset of X.

Theorem 1. *Let A, B and S be three mappings of K into itself satisfying the following conditions:*

(1.1) *S and B are W-affine,*

(1.2) *S is continuous,*

(1.3) *the pair (S,A) and (S,B) are compatible,*

(1.4) *$A(K)\subset S(K)$, $B(K)\subset S(K)$,*

(1.5) $d^p(Ax,By)\le ad^p(Sx,Sy)+b\max\left\{d^p(Ax,Sx),d^p(By,Sy)\right\}$ $+c\max\left\{d^p(Sx,Sy),d^p(Ax,Sx),d^p(By,Sy)\right\}$ *for all x, y in K, where $a,b,c>0$, $p\ge 1$, $a+b+c=1$ and* $\max\left\{\dfrac{(1-b)^2}{a},b+c\right\}<(3-3^{1-p})(3^p-1)^{-1}$.

Then A, B and S have a unique common fixed point z^* in K. also A and B are continuous at z^*.

Proof. Let $x=x_0$ be an arbitrary point in K and choose four points x_1, x_2, x_3 and x_4 in K such that

$$Sx_1=Ax,\quad Sx_2=Bx_1,\quad Sx_3=Ax_2,\quad Sx_4=Bx_3.$$

In general

$$Sx_{2r+1}=Ax_{2r},\quad Sx_{2r+2}=Bx_{2r+1},\qquad \text{for } r=0,1.$$

This can be done since $A(K)\subset S(K)$, $B(K)\subset S(K)$.

For $r=0,1$, (1.1) leads to

$$\begin{gathered}d^p(Ax_{2r},Bx_{2r-1})\le ad^p(Sx_{2r},Sx_{2r-1})+\\ b\max\left\{d^p(Ax_{2r},Sx_{2r}),d^p(Bx_{2r-1},Sx_{2r-1}\right\}+\\ +c\max\left\{d^p(Sx_{2r},Sx_{2r-1}),d^p(Ax_{2r},Sx_{2r}),d^p(Bx_{2r-1},Sx_{2r-1}\right\}.\end{gathered}$$

Therefore, we have

$$d^p(Ax_{2r},Bx_{2r-1})\le d^p(Bx_{2r-1},Sx_{2r-1}).\tag{1.6}$$

From (1.5) and (1.6), we have

$$\begin{aligned}d^p(Ax,Bx_3)&\le ad^p(Sx,Sx_3)+b\max\left\{d^p(Ax,Sx),d^p(Bx_3,Sx_3)\right\}\\ &\quad+c\max\left\{d^p(Sx,Sx_3),d^p(Ax,Sx),d^p(Bx_3,Sx_3)\right\}\\ &\le 3^pad^p(Ax,Sx)+bd^p(Ax,Sx)+3^pcd^p(Ax,Sx)\\ &=\big((a+c)3^p+b\big)d^p(Ax,Sx).\end{aligned}\tag{1.7}$$

Letting $z = W(x_2, x_4, 1/2)$ then $z \in K$ and since S is W-affine, we have

$$Sz = W(Sx_2, Sx_4, 1/2) = W(bx_1, Bx_3, 1/2). \tag{1.8}$$

It follows from (1.6)-(1.8) and $p \geq 1$ that

$$\begin{aligned} d^p(Sz, Sx_1) &= d^p(Sx_1, W(Bx_1, Bx_3, 1/2)) \\ &\leq ((1/2)d(Sx_1, Bx_1) + (1/2)d(Sx_1, Bx_3))^p \\ &\leq (1/2)(d^p(Sx_1, Bx_1) + d^p(Sx_1, Bx_3)) \\ &\leq (1/2)(1 + b + (a+c)3^p)d^p(Ax, Sx) \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} d^p(Sz, Sx_3) &= d^p(Sx_3, W(Bx_1, Bx_3, 1/2)) \\ &\leq ((1/2)d(Sx_3, Bx_1) + (1/2)d(Sx_3, Bx_3))^p \\ &\leq d^p(Ax, Sx). \end{aligned} \tag{1.10}$$

By (1.5) to (1.7), (1.10) and $p \geq 1$, we have

$$\begin{aligned} d^p(Ax, Sz) &= d^p(Ax, W(Bx_1, Bx_3, 1/2) \\ &\leq (1/2)d^p(Az, Bx_1) + (1/2)d^p(Az, Bx_3) \\ &\leq (a/4)(1 + b + (a+c)3^p)d^p(Ax, Sx) + (b/2)\max\left\{d^p(Az, Sz), d^p(Ax, Sx)\right\} + \\ &\quad (c/2)\max\left\{(1/2)(1 + b + (a+c)3^p)d^p(Ax, Sx), d^p(Az, Sz),\right. \\ &\quad \left. d^p(Ax, Sx)\right\} + (a/2)d^p(Ax, Sx) + (b/2)\max\left\{d^p(Az, Sz), d^p(Ax, Sx)\right\} \\ &\quad +(c/2)\max\left\{d^p(Ax, Sx), d^p(Az, Sz), d^p(Ax, Sz)\right\} \\ &\leq (a/4)(1 + b + (a+c)3^p + 2)d^p(Ax, Sx) \\ &\quad +b\max\left\{d^p(Az, Sz), d^p(Ax, Sx)\right\} + ((c/4)(1 + b + (a+c)3^p + (c/2)) \\ &\quad \max\left\{d^p(Az, Sz), d^p(Ax, Sx)\right\} \leq \lambda\left\{Ax, Sx), d^p(Ax, Sx)\right\}, \end{aligned} \tag{1.11}$$

where $\lambda = (a/4)(3 + b(a+c)3^p) + b + ((c/4)(1 + b + (a+c)3^p) + c/2)$. It is easy to see that $0 < \lambda < 1$ since

$$\frac{(1-b)^2}{a} < (3 - 3^{1-p})(3^p - 1)^{-1}$$

and $\lambda = \dfrac{(1-b)^2}{4}(3^p - 1) + 1$.

Hence (1.11) implies

$$d^p(Ax, Sz) \leq \lambda d^p(Ax, Sx). \tag{1.12}$$

Since x is an arbitrary point in K from (1.12) it follows that there exists a sequence $\{z_n\}$ in K such that

$$\begin{aligned} d^p(Az_0, Sz_0) &\leq \lambda d^p(Ax_0, Sx_0), \\ d^p(Az_1, Sz_1) &\leq \lambda d^p(Az_0, Sz_0), \\ d^p(Az_n, Sz_n) &\leq \lambda d^p(Az_{n-1}, Sz_{n-1}), \end{aligned}$$

which yield that $d^p(Az_n, Sz_n) \leq \lambda^{n+1} d^p(Ax_0, Sx_0)$, and so we have

$$\lim_{n\to\infty} d(Az, Sz_n) = 0, \tag{1.13}$$

Setting $K_{n_1} = \{x \in K : d(Ax, Sx) \leq 1/n_1\}$ for $n_1 = 1, 2, \ldots$ then (1.13) shows that $K_{n)1} \neq \phi$ for $n_1 = 1, 2, \ldots$ and $K_1 \supset K_2 \supset K_3 \supset \cdots$ Obviously, we have $\overline{AK_{n_1}} \neq \phi$ and $\overline{AK_{n_1}} \supset \overline{AK_{n_1+1}}$ for $n_1 = 1, 2, \ldots$

If we take $u = W(x_1, x_3, 1/2)$ and since B is W-affine we can see that

$$d^p(Au, Bu) \leq \lambda \max\big\{d^p(Ax, Sx), d^p(Ax, Sx)\big\},$$

where value of λ is same as given above. By the same way as shown above we can find

$$\lim_{n\to\infty} d(Au_n, Bu_n) = 0. \tag{1.14}$$

Seting $K_{n_2} = \{x \in K : d(Ax, Bx) \leq 1/n_2\}$ for $n_2 = 1, 2, \ldots$ then $\overline{AK_{n_2}} \neq \phi$ and $\overline{AK_{n_2}} \supset \overline{AK_{n_2+1}}$ for $n_2 = 1, 2, \ldots$

Using (1.5) and Minkowski's inequality, we have

$$\begin{aligned} d(Ax, By) \leq{}& a^{1/p} d(Sx, Sy) + b^{1/p} \max\big\{d(Ax, Sx), d(By, Sy)\big\} \\ &+ c^{1/p} \max\big\{d(Ax, Sy), d(Ax, Sx), d(By, Sy)\big\} \end{aligned} \tag{1.15}$$

for all x, y in K. for any $x, y \in K_{n_1} \cap K_{n_2}$, by (1.15), we have

$$\begin{aligned} d(Ax, By) &\leq a^{1/p} d(Sx, Sy) + (n_1^{-1} + n_2^{-1}) b^{1/p} \\ &\quad + c^{1/p} \max\big\{d(Sx, Sy), (n_1^{-1} + n_2^{-1})\big\} \\ &\leq a^{1/p}(2n_1^{-1} + d(Ax, Ay)) + (n_1^{-1} + n_2^{-1}) b^{1/p} \\ &\quad + c^{1/p} \max\big\{2n_1^{-1} + d(Ax, Ay), (n_1^{-1} + n_2^{-1})\big\}. \end{aligned}$$

Let $n = \min(n_1, n_2)$, then

$$\begin{aligned} d(Ax, By) \leq{}& a^{1/p}(2n^{-1} + d(Ax, Ay)) + 2n^{-1} b^{1/p} + \\ &+ c^{1/p} \max\big\{2n^{-1} + d(Ax, Ay)\big\}. \end{aligned} \tag{1.16}$$

Since $(a/4)(3 + b + (a + c)3^p) + b + c < \lambda < 1$, we have

$$(1/4)(b + (a + c)3^p + 3^{1-p}) < 1$$

and hence $a + c < (3 - 3^{1-p})(3^p - 1)^{-1}$. It follows from (1.15) and Lemma 3, that

$$d(Ax, By) \leq 2n^{-1}(a^{1/p} + b^{1/p} + c^{1/p})(1 - a^{1/p} - c^{1/p})^{-1}$$

Therefore, we have

$$\begin{aligned} d(Ax, Ay) &\leq d(Ax, By) + d(By, Ay) \\ &\leq 2n^{-1}(a^{1/p} + b^{1/p} + c^{1/p})(1 - a^{1/p} - c^{1/p})^{-1} + n^{-1} \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \operatorname{diam}(\overline{AK_{n_1}}) = \lim_{n\to\infty} \operatorname{diam}(AK_{n_1}) = 0.$$

Also

$$\lim_{n\to\infty} \operatorname{diam}(\overline{AK_{n_2}}) = \lim_{n\to\infty} \operatorname{diam}(AK_{n_2}) = 0.$$

By Cantor's theorem, there exists a point v_1 in K such that

$$\bigcap_{n_1=1}^{\infty} \overline{AK_{n_1}} = \{v_1\}$$

Similarly there exists a point v_2 in K such that

$$\bigcap_{n_2=1}^{\infty} \overline{AK_{n_2}} = \{v_2\}$$

So there exists a point v in K such that

$$\bigcap_{n=1}^{\infty} \overline{AK_{n}} = \{v\}.$$

Since $v \in K$ for each $n = 1, 2, \ldots$, there exists a point y_n in AK_n such that $d(y_n, v) < n^{-1}$. Then there exists a point x_n in K_n such that $d(v, Ax_n) < n^{-1}$ and so $Ax_n \to v$ as $n \to \infty$. Since $x_n \in K_n$ we also have $d(Ax_n, Sx_n) < n^{-1}$ and $d(Ax_n, Bx_n) < n^{-1}$. So $Sx_n \to v$ and $Bx_n \to v$ as $n \to \infty$. Since S is continuous, $SSx_n \to Sv$ and $SAx_n \to Sv$ as $n \to \infty$. Since S is continuous and comaptible with A by Lemma 2, we have

$$\lim_{n\to\infty} ASx_n = Sv.$$

Now, (1.15) yields that

$$\begin{aligned} d(ASxBv) \le{} & a^{1/p} d(SSx_n, Sv) + b^{1/p} \max\big\{d(ASx_n, SSx_n), d(Bv, Sv)\big\} \\ & + c^{1/p} \max\big\{d(SSx_n, Sv), d(ASx_n, SSx_n), d(Bv, Sv)\big\}. \end{aligned}$$

Letting $n \to \infty$ we obtain, $d(Sv, Bv) \le (b^{1/p} + c^{1/p}) d(Sv, Bv)$. By Lemma 3, we have $Sv = Bv$.

From (1.15) we have

$$\begin{aligned} d(Av, Bv) &\le a^{1/p} d(Sv, Sv) + b^{1/p} \max\big\{d(Av, Sv), d(Bv, Sv)\big\} \\ &\quad + c^{1/p} \max\big\{d(Sv, Sv), d(Av, Sv), d(Bv, Sv)\big\} \\ &= (b^{1/p} + c^{1/p}) d(Av, Bv) \end{aligned}$$

By Lemma 3, we have $Ab = Bv$. Then $Av = Bv = Sv$.

Thus $ASv = SAv$ and by Lemma 1, $AAv = ASv = SAv$, since S and A are compatible.

Also $BSv = SBv$, since S and B are comaptible.

Furthermore, we have

$$\begin{aligned} d^p(AAv, Bv) &\leq ad^p(SAv, Sv) + b\max\{d^p(AAv, SAv), d^p(Bv, Sv)\} \\ &+c\max\{d^p(SAv, Sv), d^p(AAv, SAv), d^p(Bv, Sv)\} \\ &= (a+c)d^p(AAv, Bv). \end{aligned}$$

This leads to $d(AAb, AV) = 0$ since $a + c < 1$.

Let $z^* = Av = Bv = Sv$. Then $Az^* = z*$ and $Sz^* = SAv = AAv = Az^* = z^*$. Also $Bz^* = BSv = SBv = SSv = Sz^* = z^*$.

Obviously, z^* is a unique common fixed point of A, B and S.

Now, we prove that A and B are continuous at z^*. Let $\{h_n\}$ be a sequence in K such that $h_n \to z^*$. Since S is continuous $Sh_n \to Sz^*$.

By (1.15), we have

$$\begin{aligned} d(Ah_n, Bz^*) &\leq a^{1/p}d(Sh_n, Sz^*) + b^{1/p}\max\{d(Ah_n, Sh_n), d(Bz^*, Sz^*)\} \\ &+c^{1/p}\max\{d(Sh_n, Sz^*), d(Ah_n, Sh_n), d(Bz^*, Sz^*)\} \\ &\leq a^{1/p}d(Sh_nSz^*) + b^{1/p}d(Ah_n, Sh_n) \\ &+c^{1/p}\max\{d(Sh_n, Sz^*), d(Ah_n, Sh_n)\}. \end{aligned}$$

Letting $n \to \infty$, we obtain $Ah_n \to Az^*$ and so A is continuous at z^*. Again by (1.15), we have

$$\begin{aligned} d(Az^*, Bh_n) &\leq a^{1/p}d(Sz^*, Sh_n) + b^{1/p}\max\{d(Az^*, Sz^*), d(Bh_n, Sh_n)\} \\ &+c^{1/p}\max\{d(Sz^*, Sh_n), d(Az^*, Sz^*), d(Bh_n, Sh_n)\}. \end{aligned}$$

Letting $n \to \infty$, we obtain $Bh_n \to Bz^*$ and so b is continuous at z^*. This completes the proof.

From Theorem 1, the following corollaries can be obtained.

Corollary 2. *Let A, B and S be three mappings of K into itself satisfying the conditions (1.1) to (1.4) of Theorem 1, and*

$$\begin{aligned} d(Ax, By) &\leq ad(Sx, Sy) + b\max\{d(Ax, Sx), d(By, Sy)\} \\ &+c\max\{d(Sx, Sy), d(Ax, Sx), d(By, Sy)\} \end{aligned} \tag{1.17}$$

for all x, y in K, where $a, b, c > 0$, $a + b + c = 1$ and $a + c < a^{1/2}$. Then A, B and S have a unique common fixed point z^* in K. Also A and B are continuous at z^*.

Corollary 3. *Let A, B and S be three mapings of K into itself satisfying the conditions (1.1) to (1.4) of Theorem 1, and*

$$d(Ax, By) \leq ad(Sx, Sy) + (1-a)\max\{d(Ax, Sx), d(By, Sy)\} \tag{1.18}$$

for all x, y in K, where $0 < a < 1$. Then A, B and S have a unique common fixed point z^* in K. Also A and B are continuous at z^*.

Remark (1) Corollary 3 is an extension of the Gregus fixed point theeorem [6] in convex meetric spaces.

(2) Theorem 1, Corollary 2 and 3 are generalization of some main results in [1], [3]-[6] and [11]-[14].

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